Chapter Review 2

1 **a** $A = X + Y + W$, so $A \sim N(8 + 12 + 15, 2 + 3 + 4)$
$A \sim N(35, 9)$

b $A = W - X$, so $A \sim N(15 - 8, 4 + 2)$
$A \sim N(7, 6)$

c $A = X - Y + 3W$, so $A \sim N(8 - 12 + 3 \times 15, 2 + 3 + 3^2 \times 4)$
$A \sim N(41, 41)$

d $A = 3X + 4W$, so $A \sim N(3 \times 8 + 4 \times 15, 3^2 \times 2 + 4^2 \times 4)$
$A \sim N(84, 82)$

e $A = 2X - Y + W$, so $A \sim N(2 \times 8 - 12 + 15, 2^2 \times 2 + 3 + 4)$
$A \sim N(19, 15)$

2 **a** $E(X - Y) = E(X) - E(Y) = 20 - 10 = 10$

b $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 5 + 4 = 9$

c Let $A = X - Y$, then $A \sim N(10, 9)$
$P(13 < X - Y < 16) = P(A < 16) - P(A < 13) = 0.9772 - 0.8413 = 0.1359$ (4 d.p.)

3 **a** $E(R) = E(X) + 4E(Y) = 8 + (4 \times 14) = 64$

b $\text{Var}(R) = \text{Var}(X) + 16\,\text{Var}(Y) = 2^2 + (16 \times 3^2) = 148$

c $R \sim N(64, 148)$, $P(R < 41) = 0.0293$ (4 d.p.)

d $S = Y_1 + Y_2 + Y_3 - 0.5X$
$\text{Var}(S) = 3\,\text{Var}(Y) + \left(\tfrac{1}{2}\right)^2 \text{Var}(X) = 3 \times 9 + \tfrac{1}{4} \times 4 = 27 + 1 = 28$

4 **a** **i** Let PB be the thickness of a randomly selected paperback and HB be the thickness of a randomly selected hardback, then $PB \sim N(2.1, 0.39)$ and $HB \sim N(4.0, 1.56)$
Let Y be the thickness of 15 randomly selected paperbacks, $Y = PB_1 + PB_2 + PB_3 + \ldots + PB_{15}$
$E(Y) = 15 \times 2.1 = 31.5$ $\quad$ $\text{Var}(Y) = 15 \times 0.39 = 5.85$
So $Y \sim N(31.5, 5.85)$
$P(Y < 30) = 0.2676$ (4 d.p.)

4 a ii Let Z be the thickness of 5 randomly selected paperbacks and 5 randomly selected hardbacks, then $Z = PB_1 + PB_2 + PB_3 + PB_4 + PB_5 + HB_1 + HB_2 + HB_3 + HB_4 + HB_5$

$\text{E}(Z) = 5 \times 2.1 + 5 \times 4.0 = 30.5 \qquad \text{Var}(Z) = 5 \times 0.39 + 5 \times 1.56 = 9.75$

So $Z \sim \text{N}(30.5, 9.75)$

$\text{P}(Z < 30) = 0.4364$ (4 d.p.)

b Using $Y \sim \text{N}(31.5, 5.85)$ from part **ai**, find x such that $\text{P}(Y < x) = 0.99$

Using the inverse normal distribution function of the calculator, $x = 37.1$ cm (3 s.f.).

5 a Let A be the difference in mass of two randomly selected Yummies, so $A = Y_1 - Y_2$.

$\text{E}(A) = \text{E}(Y_1) - \text{E}(Y_2) = 0 \qquad \text{Var}(A) = \text{Var}(Y_1) + \text{Var}(Y_2) = 32$

So $A \sim \text{N}(0, 32)$

Required to find $\text{P}(A > 5) + \text{P}(A < -5) = 1 - \text{P}(A < 5) + \text{P}(A < -5) = 0.3768$ (4 d.p.)

b Let $B = Y - X$, then $B \sim \text{N}(32 - 30, 16 + 25)$, so $B \sim \text{N}(2, 41)$

Then $\text{P}(B > 0) = 1 - \text{P}(B < 0) = 1 - 0.3774 = 0.6226$ (4 d.p.)

c Let Z be the thickness of 6 randomly selected Xtras and 4 randomly selected Yummies.

$\text{E}(C) = 6 \times 30 + 4 \times 32 = 308 \qquad \text{Var}(A) = 6 \times 25 + 4 \times 16 = 214$

So $C \sim \text{N}(308, 214)$

$\text{P}(280 < C < 330) = \text{P}(C < 330) - \text{P}(C < 280) = 0.9337 - 0.0278 = 0.9059$ (4 d.p.)

6 Let B be the mass of a randomly selected biscuit, W be the mass of an individual wrapper, M be the mass of the packaging material and A be total mass of a packet of 6 biscuits, then:

$\text{E}(A) = 6 \times 75 + 6 \times 10 + 40 = 550 \qquad \text{Var}(A) = 6 \times 5^2 + 6 \times 2^2 + 3^2 = 183$

So $A \sim \text{N}(550, 183)$

$\text{P}(535 < A < 565) = \text{P}(A < 565) - \text{P}(A < 535) = 0.8662 - 0.1337 = 0.732$ (3 d.p.)

7 a i $\text{E}(Q) = 2\text{E}(X) + \text{E}(Y) = 2 \times 10 + 40 = 60$

ii $\text{Var}(Q) = 2^2\,\text{Var}(X) + \text{Var}(Y) = 2^2 \times 2^2 + 3^2 = 25$

b i $\text{E}(R) = 5\text{E}(X) = 5 \times 10 = 50$

$\text{Var}(R) = 5 \times \text{Var}(X) = 5 \times 2^2 = 20$

So $R \sim \text{N}(50, 20)$

ii Let $S = Q - R$, so $S \sim \text{N}(60 - 50, 25 + 20)$, i.e. $S \sim \text{N}(10, 45)$S

$\text{P}(Q > R) = \text{P}(Q - R > 0) = 1 - \text{P}(S < 0) = 1 - 0.0680 = 0.9320$ (4 d.p.)

8 a Let C be the usable capacity of a randomly selected games console, G be the storage required by a randomly selected game and A be storage required by 10 games, then:

$C \sim \text{N}(60, 2.5^2) \qquad G \sim \text{N}(5.5, 1.2^2) \qquad A \sim \text{N}(10 \times 5.5, 10 \times 1.2^2) \Rightarrow A \sim \text{N}(55, 14.4)$

Let $B = C - A$, so $B \sim \text{N}(60 - 55, 14.4 + 6.25) \Rightarrow B \sim \text{N}(5, 20.65)$

Required to find $\text{P}(B > 0) = 1 - \text{P}(B < 0) = 1 - 0.1356 = 0.8644$ (4 d.p.)

8 b Assuming that all random variables are independent, i.e. that the storage space required by each game and the usable capacity of the console are all independent.

9 $Y \sim \text{N}(3\times4, 3\times0.03)$, so $Y \sim \text{N}(12, 0.09)$

$Z \sim \text{N}(3\times4, 3^2\times0.03)$, so $Z \sim \text{N}(12, 0.27)$

Let $W = Z - Y$, so $W \sim \text{N}(12-12, 0.27+0.09) \Rightarrow W \sim \text{N}(0, 0.36)$

Required to find $\text{P}(-1 < W < 1) = \text{P}(W < 1) - \text{P}(W < -1) = 0.9522 - 0.0478 = 0.9044$ (4 d.p.)

10 a $L \sim \text{N}(75, 5^2)$, $\text{S} \sim \text{N}(40, 3^2)$

Let $D = S - 0.5L$, so $D \sim \text{N}(40 - 0.5\times75, 3^2 + 0.5^2\times5^2)$

So $D \sim \text{N}(2.5, 15.25)$

$\text{P}(D > 0) = 1 - \text{P}(D < 0) = 1 - 0.2610 = 0.7390$ (4 d.p.)

b $M \sim \text{N}(10\times40, 10\times3^2)$, so $M \sim \text{N}(400, 90)$

$\text{P}(|M - 400| < 5) = \text{P}(395 < M < 405) = \text{P}(M < 405) - \text{P}(M < 395)$

$= 0.7019 - 0.2981 = 0.4038$ (4 d.p.)

Challenge

$$\begin{aligned}
\text{Var}(X+Y) &= \text{E}\left((X+Y)^2\right) - \left(\text{E}(X+Y)\right)^2 \\
&= \text{E}(X^2 + 2XY + Y^2) - \left(\text{E}(X)+\text{E}(Y)\right)\left(\text{E}(X)+\text{E}(Y)\right) \\
&= \text{E}(X^2) + 2\text{E}(XY) + \text{E}(Y^2) - \left(\text{E}(X)\text{E}(X) + 2\text{E}(X)\text{E}(Y) + \text{E}(Y)\text{E}(Y)\right) \\
&= \text{E}(X^2) - \text{E}(X)\text{E}(X) + 2\text{E}(XY) - 2\text{E}(X)\text{E}(Y) + \text{E}(Y^2) - \text{E}(Y)\text{E}(Y) \\
&= \text{E}(X^2) - (\text{E}(X))^2 + \text{E}(Y^2) - (\text{E}(Y))^2 \qquad \text{as } \text{E}(XY) = \text{E}(X)\text{E}(Y) \\
&= \text{Var}(X) + \text{Var}(Y)
\end{aligned}$$

